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# Fractional- and integer-order moments, and multiscaling for smoothly truncated Lévy flights

Gy. Terdik [a,1], W.A. Woyczynski [b,*,2], A. Piryatinska [c]

[a] *Department of Information Technology, University of Debrecen, 4010, Hungary*
[b] *Department of Statistics and Center for Stochastic and Chaotic Processes in Science and Technology, Case Western Reserve University, Cleveland, OH 44106, USA*
[c] *Jet Propulsion Laboratory, California Institute of Technology, Pasadena, CA 91125, USA*

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**Abstract**

We study multiscale properties of smoothly truncated Lévy flights. The behavior of both fractional- and integer-order moments $\langle|X_a(t)|^\rho\rangle$, for both small and large values of the scaling parameter $a$ is investigated. In the former case we obtain the behavior close to that of the $\alpha$-stable flight, and for the latter, close to that of the Brownian motion.


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## 1. Introduction

Truncated Lévy flights were introduced by Mantegna and Stanley [8] as a model for random phenomena which exhibit at small scales properties similar to those of selfsimilar Lévy processes (see, e.g., [14]) but have distributions which at large scales have cutoffs and thus have finite moments of any order. Mantegna and Stanley pointed out the usefulness of truncated Lévy flights in modeling a broad spectrum of random phenomena ranging from turbulence

* Corresponding author.
*E-mail address:* waw@cwru.edu (W.A. Woyczynski).
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to financial markets. Smoothly truncated Lévy flights (STLF) introduced by Koponen [7], built on Mantegna and Stanley's ideas but stressed the advantage of a nice analytic form. To be more precise let us recall that the 1D STLF $X(t)$ is a Lévy process, that is a process with homogeneous and independent increments and $X(0)=0$, with the one-point probability distribution of $X(1)$ having characteristic function of the form

$$\varphi_{X(1)}(u)=\exp\bigl(\psi_{X(1)}(u)\bigr),$$

where the *cumulant function*

$$\psi_{X(1)}(u)=a\lambda^{\alpha}\bigl[p\zeta_{\alpha}(-u/\lambda)+q\zeta_{\alpha}(u/\lambda)\bigr]+iub,$$

where $\lambda>0$, $a,p,q\geqslant 0$, $p+q=1$, $b$ is a real number, and

$$\zeta_{\alpha}(r)=\begin{cases}\Gamma(-\alpha)[(1-ir)^{\alpha}-1], & \text{for } 0<\alpha<1,\\ (1-ir)\log(1-ir)+ir, & \text{for } \alpha=1,\\ \Gamma(-\alpha)[(1-ir)^{\alpha}-1+i\alpha r], & \text{for } 1<\alpha<2.\end{cases}$$

In what follows we shall only consider, without loss of generality, the case when the shift parameter $b=0$. Parameters $p$ and $q$ describe the *skewness* of the probability distributions, and $p=q=1/2$ yields a symmetric distribution. Parameter $\lambda$ will be referred to as the *truncation* parameter.

In the case of $0<\alpha<1$, the cumulant function is given by the formula

$$\psi_{X(1)}(u)=a\lambda^{\alpha}\Gamma(-\alpha)\left[p\left(1+i\frac{u}{\lambda}\right)^{\alpha}+q\left(1-i\frac{u}{\lambda}\right)^{\alpha}-1\right], \tag{1}$$

and if $p=0$ the cumulant function

$$\psi_{X(1)}(u)=a\lambda^{\alpha}\Gamma(-\alpha)\left[\left(1-i\frac{u}{\lambda}\right)^{\alpha}-1\right]=a\Gamma(-\alpha)\bigl[(\lambda-iu)^{\alpha}-\lambda^{\alpha}\bigr], \tag{2}$$

describes a distribution totally concentrated on the positive half-line.

In the symmetric case $p=q=1/2$, and $0<\alpha<2, \alpha\neq 1$,

$$\begin{aligned}\psi_{X(1)}(u)&=a\lambda^{\alpha}\Gamma(-\alpha)\left[\frac{1}{2}\left(1+i\frac{u}{\lambda}\right)^{\alpha}+\frac{1}{2}\left(1-i\frac{u}{\lambda}\right)^{\alpha}-1\right]\\ &=a\Gamma(-\alpha)\bigl[\bigl(\lambda^{2}+u^{2}\bigr)^{\alpha/2}\cos\bigl(\alpha\arctan(u/\lambda)\bigr)-\lambda^{\alpha}\bigr].\end{aligned} \tag{3}$$

If $\alpha=1$ in the symmetric case the cumulant function is

$$\psi_{X(1)}(u)=a\left(\frac{\lambda}{2}\log\left(1+\frac{u^{2}}{\lambda^{2}}\right)-|u|\arctan\frac{|u|}{\lambda}\right). \tag{4}$$

Thus, in this Letter, the STLF $X(t)$ is uniquely determined by the index $\alpha$, scale parameter $a$, skewness parameter $p$ and truncation parameter $\lambda$. The distribution of $X(1)$ will be denoted $STLF_{\alpha}(a,p,\lambda)$. The index $\alpha$ corresponds to the nontruncated limit when $\lambda=0$, in this case the distribution of $X(1)$ is the classical Lévy's $\alpha$-stable probability distribution. The scale parameter $a$ tunes the time unit to $a$, the distribution of $X(t)$ is then $STLF_{\alpha}(at,p,\lambda)$.

The role of the truncation parameter $\lambda$ is obvious in the following particular case. It is worthwhile observing that if $f(x)$ is a probability density function (PDF) of a totally asymmetric $\alpha$-stable distribution $STLF_{\alpha}(a,0,0)$, where $0<\alpha<1$, (see, (2)) then

$$g_{\lambda}(x):=f(x)e^{-\lambda|x|}$$

is, up to a normalizing constant, the PDF of $STLF_{\alpha}(a,0,\lambda)$. Vice versa, if $g_{\lambda}(x)$ is a totally asymmetric PDF of $STLF_{\alpha}(a,0,\lambda)$ then, up to a normalizing constant, $f(x):=g_{\lambda}(x)e^{\lambda|x|}$, is an $\alpha$-stable density function. This transformation, i.e., the multiplication by $e^{-\lambda|x|}$ does not change the behavior of the $\alpha$-stable density, but it makes the tail of the distribution thinner. This peculiar fact has been observed, for $\alpha = 1/2$, by Bochner (see, e.g., [2], and [14, p. 233], for the general case), and the related transformations are often called subordination, and Esscher or exponential, transformation. The calculations in this case are straightforward. Indeed, the PDF of $STLF_{1/2}(a, 0, 0)$ is explicitly known and of the form

$$f(x) = \frac{a}{x^{3/2}} e^{-a^2\pi/x}, \quad x > 0,$$

with the cumulant function

$$\psi(u) = -2a\sqrt{\pi}(-iu)^{1/2}.$$

The distribution $STLF_{1/2}(a, 0, 0)$ is known as the *inverse Gaussian* (*IG*) distribution (see, [15]). For any positive $\lambda$, the corresponding $STLF_{1/2}(a, 0, \lambda)$ produces an exponential family $IG(\lambda)$ generated by the *IG* distributions with PDFs

$$g_\lambda(x) = \frac{ae^{-2a\sqrt{\pi\lambda}}}{x^{3/2}} e^{-a^2\pi/x-\lambda x}, \quad x > 0,$$

with cumulant function

$$\psi_\lambda(u) = -2a\sqrt{\pi}\lambda^{1/2}\left[\left(1 - i\frac{u}{\lambda}\right)^{1/2} - 1\right].$$

Recently, there have been several attempts to generalize the above transformation. In particular, gradually truncated Lévy flights (GTLF) have been studied by [6], and the exponentially damped Lévy flights (EDLF) in [5]; both papers contain further valuable references. The above mentioned generalizations are based on specific choices of truncation schemes for the symmetric $\alpha$-stable density $f(\cdot)$. In the case of GTLF the multiplier

$$\xi(Z_{\Delta t}) = \begin{cases} 1, & \text{if } |Z_{\Delta t}| \leqslant l_c, \\ \exp\left[-\left(\frac{|Z_{\Delta t}|-l_c}{k}\right)^{\beta_0}\right], & \text{if } |Z_{\Delta t}| > l_c, \end{cases}$$

is utilized, while, in the case of EDLF, the truncation factor is of the form

$$\xi(Z_{\Delta t}) = \begin{cases} 1, & \text{if } |Z_{\Delta t}| \leqslant l_c, \\ (\Delta t^{-1/\alpha}|Z_{\Delta t}| + \eta)^{\beta_1} \exp(H(Z_{\Delta t})), & \text{if } l_c \leqslant |Z_{\Delta t}| < l_{\max}, \\ 0, & \text{if } l_{\max} \leqslant |Z_{\Delta t}|, \end{cases}$$

where $H(Z_{\Delta t}) = \lambda_1 + \lambda_2(1 - |Z_{\Delta t}|/l_{\max})^{\beta_2} + \lambda_2(|Z_{\Delta t}| - l_c)^{\beta_3}$. These truncations permit a flexible model fitting given empirical marginal distributions.

In the present Letter we take a closer look at the scaling properties of STLFs following on the footsteps of Nakao [10]. The typical approach, used commonly in, e.g., intermittency studies in hydrodynamics (see [4,9]), calls for an investigation of evolution of moments (partition functions) of STLFs as functions of time and scale, displayed in the log–log coordinates. In contrast to Nakao's work, we study the behavior of both fractional- and integer-order moments $\langle |X_a(t)|^\rho \rangle$, for both small and large values of the scaling parameter $a$. In the former case, for a fixed $t$, we obtain the behavior close to that of the $\alpha$-stable distribution, and for the latter, close to that of the Gaussian distribution (Sections 2 and 3).

More precisely, we will consider an STLF $X(t)$, with discrete time parameter $t = 0, 1, 2, 3, \ldots$. To emphasize the dependence of STLF on the scaling parameter $a$ we will write $X(t) = X_a(t)$ and calculate the shape of the *scaling exponent* $\tau_a(\rho)$ such that

$$\left\langle \left|X_a(t)\right|^\rho \right\rangle \propto (at)^{\tau_a(\rho)},$$

which gives the following linear dependence in the log–log scales:

$$\log\left\langle \left|X_a(t)\right|^\rho \right\rangle \approx \tau_a(\rho) \log(at) + \text{const},$$

where the behavior of $\tau_a(\rho)$ is different for small and large values of the scaling parameter $a$, see Section 4.

## 2. Fractional- and integer-order moments of one-sided $STLF_\alpha$ distributions, $0 < \alpha < 1$

### 2.1. Fractional-order moments

For the one-sided $STLF_\alpha(a, 0, \lambda)$ distribution with $0 < \alpha < 1$, the cumulant function is of the form

$$\psi_{X_a}(u) = a\lambda^\alpha \Gamma(-\alpha)\left[\left(1 - i\frac{u}{\lambda}\right)^\alpha - 1\right]. \tag{5}$$

As $\lambda \to 0$, the distribution $STLF_\alpha(a, 0, \lambda)$ clearly converges to the $\alpha$-stable distribution $STLF_\alpha(a, 0, 0)$, for which the fractional moments of order $\rho < \alpha$ are well known and of the form

$$\langle |X_a|^\rho \rangle = \bigl(-a\Gamma(-\alpha)\bigr)^{\rho/\alpha} \frac{\Gamma(1-\rho/\alpha)}{\Gamma(1-\rho)}, \quad 0 < \rho < \alpha, \tag{6}$$

see, e.g., [18], or [14, p. 162].

In this subsection we address the question of dependence of the fractional moments of $STLF_\alpha(a, 0, \lambda)$ on scaling parameter $a$, for a fixed $\lambda$. Recall that random variables involved here are positive and the absolute moments and moments of the same order coincide.

Since the $STLF_\alpha(a, 0, \lambda)$ distribution is concentrated on the positive half-line, it is more convenient to use its Laplace transform instead of the Fourier transform. The former is given by the formula

$$\phi_{X_a}(u) = \langle e^{uX_a} \rangle = \varphi_{X_a}(-iu) = \exp\bigl(a\lambda^\alpha \Gamma(-\alpha)\bigl[(1 + u/\lambda)^\alpha - 1\bigr]\bigr),$$

so that

$$\langle |X_a|^\rho \rangle = \frac{\rho}{\Gamma(1-\rho)} \int_0^\infty \bigl[1 - \exp\bigl(a\lambda^\alpha \Gamma(-\alpha)\bigl[(1+u/\lambda)^\alpha - 1\bigr]\bigr)\bigr] \frac{du}{u^{1+\rho}},$$

see [18]. Integrating by parts, and changing variables, we get

$$\langle |X_a|^\rho \rangle = \frac{a\lambda^{\alpha-\rho}\Gamma(1-\alpha)}{\alpha\Gamma(1-\rho)} \int_1^\infty \frac{\exp(a\lambda^\alpha \Gamma(-\alpha)[u-1])}{(u^{1/\alpha}-1)^\rho}\, du, \quad 0 < \rho < \alpha. \tag{7}$$

#### 2.1.1. Fractional-order moments in the case $\alpha = 1/2$

Observe that only in the particular case $\alpha = 1/2$ the above formulas lead to explicit expressions. Indeed, in this case,

$$\langle |X_a|^\rho \rangle = \frac{2a\lambda^{1/2-\rho}\sqrt{\pi}}{\Gamma(1-\rho)} \int_1^\infty \frac{\exp(-2a\sqrt{\pi\lambda}[u-1])}{(u^2-1)^\rho}\, du, \quad 0 < \rho < 1/2,$$

and the fractional moments of $STLF_{1/2}(a, 0, \lambda)$ distribution, are

$$\langle |X_a|^\rho \rangle = 2a^{1/2+\rho}\left(\frac{\lambda}{\pi}\right)^{1/4-\rho/2} \exp(2a\sqrt{\pi\lambda}) K_{1/2-\rho}(2a\sqrt{\pi\lambda}), \qquad 0 < \rho < 1/2, \tag{8}$$

where $K_\nu$ is the modified Bessel function of the second kind, see [11, p. 323, 2.3.5.4]. Since,

$$K_\nu(z) \sim \frac{1}{2}\Gamma(\nu)\left(\frac{z}{2}\right)^{-\nu}, \quad z \to 0,$$

see [1, 9.6.9], we obtain the following asymptotics for the moments of $STLF_\alpha$:

$$\langle |X_a|^\rho\rangle \sim 2a^{1/2+\rho}\left(\frac{\lambda}{\pi}\right)^{1/4-\rho/2}\frac{1}{2}\Gamma(1/2-\rho)(a\sqrt{\pi\lambda})^{\rho-1/2} = \bigl(-a\Gamma(-1/2)\bigr)^{2\rho}\frac{\Gamma(1-2\rho)}{\Gamma(1-\rho)}, \quad a\to 0,$$

so, for a fixed $\lambda$, at small-time scales $a$, the moments structure of the $STLF_{1/2}$ distribution is the same as that of the $1/2$-stable distribution, see (6).

Since

$$K_\nu(z) = \left(\frac{\pi}{2z}\right)^{1/2} e^{-z}\bigl(1+O(1/z)\bigr), \quad z\to\infty,$$

(see Erdélyi [3, vol. 2, p. 32]), we have, for large time-scales $a\to\infty$, the asymptotics

$$\langle |X_a|^\rho\rangle \sim a^\rho(\pi/\lambda)^{\rho/2}, \quad a\to\infty,\ 0<\rho<1/2,$$

which gives the following logarithmic asymptotics for the fractional moments:

$$\log\langle |X|^\rho\rangle \sim \rho\log a + \text{const}, \quad a\to\infty.$$

*2.1.2. Fractional-order moments for general $\alpha$, $0<\alpha<1$*

We shall return now to the general case of $0<\alpha<1$. Formula (7) can be rewritten in the form

$$\langle |X_a|^\rho\rangle = \frac{\Gamma(1-\alpha)a\lambda^{\alpha-\rho}}{\alpha\Gamma(1-\rho)}\int_1^\infty \frac{\exp(a\lambda^\alpha\Gamma(-\alpha)[u-1])}{(u-1)^\rho}\left[\frac{u-1}{u^{1/\alpha}-1}\right]^\rho du.$$

Function $f(u) := (u-1)/(u^{1/\alpha}-1)$ has the limit $\alpha$ for $u\to 1$, and the limit 0 at $u\to\infty$, with the Taylor series expansion around $u=1$ of the form

$$f(u) = \alpha - \frac{1-\alpha}{2}(u-1) + o\bigl([u-1]\bigr), \tag{9}$$

and the expansion around $u=\infty$ of the form

$$f(u) = \frac{1}{u^{1/\alpha-1}} - \frac{1-\alpha}{\alpha u^{1/\alpha}} + o\left(\frac{1}{u^{1/\alpha}}\right). \tag{10}$$

Therefore,

(1) For small $a$, or, equivalently, large $u/(a\lambda^\alpha\Gamma(-\alpha))$, expansion (10) applied to the integrand in

$$\langle |X|^\rho\rangle = a^\rho\frac{(-\Gamma(-\alpha))^\rho}{\Gamma(1-\rho)}\lambda^{(\alpha-1)\rho}\int_0^\infty\frac{\exp(-u)}{u^\rho}\left[\frac{u/(a\lambda^\alpha\Gamma(-\alpha))}{(u/(a\lambda^\alpha\Gamma(-\alpha))+1)^{1/\alpha}-1}\right]^\rho du,$$

results in asymptotics

$$\begin{aligned}\langle |X|^\rho\rangle &= a^\rho\frac{(-\Gamma(-\alpha))^\rho}{\Gamma(1-\rho)}\lambda^{(\alpha-1)\rho}\int_0^\infty\frac{\exp(-u)}{u^\rho}\left[\frac{(a\lambda^\alpha\Gamma(-\alpha))^{1/\alpha-1}}{u^{1/\alpha-1}} + O\left(\frac{1}{u^{1/\alpha-1}}\right)\right]^\rho du\\ &\sim a^{\rho/\alpha}\frac{(-\Gamma(-\alpha))^{\rho/\alpha}\Gamma(1-\rho/\alpha)}{\Gamma(1-\rho)}, \quad a\to 0,\ 0<\rho<\alpha,\end{aligned}$$

which again matches the moment behavior for the $\alpha$-stable case (6).

(2) For large $a$, or, equivalently, small $u/(a\lambda^\alpha\Gamma(-\alpha))$, expansion (9) applies to yield asymptotics

$$\langle |X|^\rho \rangle \sim a^\rho \big(\Gamma(1-\alpha)\big)^\rho \lambda^{(\alpha-1)\rho}, \quad a \to \infty,\ 0 < \rho < \alpha.$$

In conclusion, for each $\alpha$, $0 < \alpha < 1$, and each $\rho < \alpha$,

$$\log\langle |X|^\rho \rangle \sim \begin{cases} (\rho/\alpha)\log a + c_1, & \text{as } a \to 0, \\ \rho \log a + c_2, & \text{as } a \to \infty, \end{cases}$$

where constants $c_1 = \log\,(-\Gamma(-\alpha))^{\rho/\alpha}\Gamma(1-\rho/\alpha)/\Gamma(1-\rho)$, and $c_2 = \rho\log(\lambda^{\alpha-1}\Gamma(1-\alpha))$.

### *2.2. Integer-order moments*

For $\lambda > 0$, the $STLF_\alpha(a, 0, \lambda)$ distribution has finite moments of any positive order $\rho$, the fact crucial for their application as models of the physical and economic phenomena (see, e.g., Mantegna and Stanley [8]). In this subsection we will show how to explicitly calculate all the positive integer moments of $STLF_\alpha(a, 0, \lambda)$ by using the theory of cumulants. This method will be applied in the case of $\rho = 1, 2$, and 3.

Recall that the $k$th *cumulant* of any random quantity $Z$ is defined as the value at $u = 0$ of the $k$th derivative of the cumulant function of $Z$ (the logarithm of the characteristic function of $Z$):

$$\mathrm{cum}_k(Z) = (-i)^k \frac{d^k}{du^k}\psi_Z(u)\bigg|_{u=0},$$

see, e.g., [17].

In view of (5), the cumulants of order $k = 1, 2, \ldots,$ of $STLF_\alpha(a, 0, \lambda)$ are given by the expression

$$\mathrm{cum}_k(X_a) = a\lambda^{\alpha-k}\Gamma(k-\alpha), \quad k = 1, 2, \ldots.$$

For a random quantity $Z$ the integer-order moments of $Z$ can be expressed in terms of its cumulants as follows:

$$\begin{aligned} \langle Z \rangle &= \mathrm{cum}_1(Z), \\ \langle Z^2 \rangle &= \mathrm{cum}_2(Z) + \big[\mathrm{cum}_1(Z)\big]^2, \\ \langle Z^3 \rangle &= \mathrm{cum}_3(Z) + 3\mathrm{cum}_1(Z)\mathrm{cum}_2(Z) + \big[\mathrm{cum}_1(Z)\big]^3, \\ &\ \ \vdots \\ \langle Z^k \rangle &= \mathrm{cum}_k(Z) + \cdots + \big[\mathrm{cum}_1(Z)\big]^k, \end{aligned}$$

and so on.

For a fixed $k$, and large $a$, the term $[\mathrm{cum}_1(X_a)]^k$ dominates. Hence we conclude that

$$\log\langle |X_a|^k \rangle \sim k\log a + c, \quad \text{as } a \to \infty, \tag{11}$$

where $c$ is a constant independent of $a$. As we will see in Section 4 this is the first evidence of the multiscaling property of $STLF_\alpha(a, 0, \lambda)$ flights, and reflects its large time-scale Gaussian (with drift) behavior.

For a fixed $k$, and as $a \to 0$, the dominant term is $\mathrm{cum}_k(X_a)$. Therefore

$$\log\langle |X_a|^k \rangle \sim \log a + c, \quad \text{as } a \to 0.$$

At this time we are not providing an evaluation of theoretical fractional moments of order $\rho, \alpha \leqslant \rho < \infty$. However, the method used in this Section can be extended to provide information about any fractional nonnegative moment of order $\rho$, producing the formula

$$\log\langle |X|^\rho \rangle \sim \begin{cases} \min(\rho/\alpha, 1)\log a + c_1, & \text{as } a \to 0, \\ \rho \log a + c_2, & \text{as } a \to \infty, \end{cases}$$

which is also supported by our simulations in Section 4. There, we will simulate the $STLF_\alpha$ distribution and calculate the behavior of $\log\langle|X_a|^\rho\rangle$ via a sampling procedure. In the special case $\alpha = 1/2$, formula (8) remains valid for any positive $\rho$ giving us complete information about the behavior of moments.

## 3. Fractional- and integer-order moments for symmetric $STLF_\alpha$ $0 < \alpha < 2$

### 3.1. Fractional-order moments

As seen in $STLF_\alpha(at, p, 0)$ is which, in the symmetric case $p = q = 1/2$, has the characteristic function of the following simple form

$$\exp\bigl(-ta\Gamma(-\alpha)\cos(\alpha\pi/2)|u|^\alpha\bigr), \quad 0 < \alpha < 2.$$

As seen in (3) and (4), in the case of symmetric $STLF_\alpha(a, 1/2, \lambda)$ distribution the characteristic function $\varphi_{X_a}(u) = \exp(\psi_{X_a}(u))$ has the cumulant function of the explicit form

$$\psi_{X_a}(u) = \begin{cases} a\Gamma(-\alpha)\lambda^\alpha\bigl[\bigl(1+\frac{u^2}{\lambda^2}\bigr)^{\alpha/2}\cos(\alpha\arctan(u/\lambda)) - 1\bigr], & \text{for } \alpha \neq 1, \\ a\bigl(\frac{1}{2}\lambda\log\bigl(1+\frac{u^2}{\lambda^2}\bigr) - |u|\arctan(|u|/\lambda)\bigr), & \text{for } \alpha = 1, \end{cases} \tag{12}$$

therefore $\psi_{X_a}$ is real-valued. In spite of this fact, it is hard to replicate for them direct calculations of asymptotics of fractional moments used in Section 2 in the case of one-sided STLF distributions. So, to investigate the behavior of absolute moments of $STLF_\alpha(a, 1/2, \lambda)$ we will have to be satisfied with just finding asymptotics for their upper and lower bounds. The integer-order moment calculations are similar to those in Section 2.2.

The discussion in Appendix A, Section A.1 can be summarized as follows:

$$\log\bigl\langle|X|^\rho\bigr\rangle \sim \begin{cases} (\rho/\alpha)\log a + C_1, & \text{for } a \to 0, \\ (\rho/2)\log a + C_2, & \text{for } a \to \infty, \end{cases}$$

for some, independent of $a$, constants $C_1, C_2$.

Thus the asymptotic dependence of the log-fractional absolute moments on $\log a$ is linear, both, for short and large time-scales $a$. As we shall see in Section 4, for time-scales $a \to 0$, a symmetric $STLF_\alpha$ flight converges to the $\alpha$-stable Lévy flight, and, for time-scales $a \to \infty$, it converges to the increments of the Brownian motion. This result gives the logarithmic asymptotics indicated above.

### 3.2. Integer-order moments

For $STLF_\alpha(a, 1/2, \lambda)$, $\alpha \neq 1$, the cumulant technique used in Section 2.2 permits only evaluation of absolute moments of even order $\rho = 2n$. Other moments can be obtained via the Central Limit Theorem as is done in Section 4. The calculations analogous to those of Section 2.2 give the following logarithmic asymptotics.

For a fixed $k$, and large $a$,

$$\log\bigl\langle|X_a|^k\bigr\rangle \sim \frac{k}{2}\log a + c, \quad \text{as } a \to \infty,$$

where $c$ is a constant independent of $a$. As we will see in Section 4 this is the first evidence of the multiscaling property of $STLF_\alpha(a, 0, \lambda)$ flights, and reflects its large time-scale Gaussian (no drift) behavior.

For example, if $2n = 4$, then

$$\bigl\langle X^4\bigr\rangle = \mathrm{cum}_4(X) + 3\bigl[\mathrm{cum}_2(X)\bigr]^2.$$

Since, for any symmetric distribution $STLF_\alpha(a, 1/2, \lambda)$,

$$\mathrm{cum}_{2k}(X) = a\lambda^{\alpha-2k}\Gamma(2k-\alpha),$$

we have the asymptotics $\langle X^4\rangle \sim a^2$, as $a \to \infty$.

For small $a$ we have that

$$\log\langle |X_a|^k\rangle \sim \log a + c, \quad \text{as } a \to 0.$$

We conclude that there is no difference between the behavior of fractional and integer moments either $a$ is large or small.

## 4. Multiscaling of the smoothly truncated Lévy flight

Consider now the $STLF_\alpha$ flight $X(t) = X_a(t)$ determined by cumulant function

$$\psi_{X_a(t)}(u|\lambda) = ta\lambda^\alpha\Gamma(-\alpha)\left[p\left(1+i\frac{u}{\lambda}\right)^\alpha + q\left(1-i\frac{u}{\lambda}\right)^\alpha - 1\right],$$

see (1).

Our basic contention, grounded in moment calculations from Sections 2 and 3, is that two different rescalings of $STLF_\alpha$ give two different behaviors for large and small $t$'s: the rescaling

$$a^{-1/\alpha}X_a(t) \Rightarrow S_\alpha(t), \quad a \to 0, \tag{13}$$

where the convergence $\Rightarrow$ is that of one-point distributions, gives the behavior of the $\alpha$-stable process $S_\alpha(t)$, while the rescaling

$$a^{-1/2}\big(X_a(t) - \langle X_a(t)\rangle\big) \Rightarrow B(t), \quad a \to \infty, \tag{14}$$

gives the behavior of the Brownian motion process $B(t)$.

We will argue this multiscaling statement by looking at the asymptotics of the logarithmic partition functions

$$\log\langle\left|a^{-1/\alpha}X_a(t)\right|^\rho\rangle = -\frac{\rho}{\alpha}\log a + \log\langle\left|X_a(t)\right|^\rho\rangle, \quad a \to 0,$$
$$-\frac{\rho}{\alpha}\log a + \frac{\rho}{\alpha}\log(at) + c = \frac{\rho}{\alpha}\log t + c,$$

for $\rho < \alpha$, where the behavior of the last term is known from Sections 1 and 2. We concern here integer moments only (it is enough for the CLT). The central moments of even orders ($\rho = 2n$) for any $\alpha \in (0, 2]$ fulfills

$$\log\langle a^{-1/2}\big(X_a(t) - \langle X_a(t)\rangle\big)^\rho\rangle \to \frac{\rho}{2}\log t + c, \quad a \to \infty,$$

and it is easy to see that the central moments of odd orders are either zero, ($\alpha \in (1, 2]$, symmetric case) or $a^{1/2-\varepsilon}$ in magnitude, as $a \to \infty$. Therefore Central Limit Theorem provides the limit distribution of

$$a^{-1/2}\big(X_a(t) - \langle X_a(t)\rangle\big),$$

it is Gaussian with mean zero and variance $\mathrm{Var}(X_1(t)) \propto t$ so that its $\rho$th central moments of even orders grow like $t^{\rho/2}$.

Since the characteristic functions of the processes in question are analytic the convergence of the corresponding logarithmic partition functions implies the convergence of processes in (13) and (14). A different approach to this problem, in a more general context of *tempered stable processes* has been announced by Rosiński [12]. It is our understanding that the full version of his results will be submitted for publication soon, see [13].

### 4.1. One-sided $STLF_\alpha$ flights, $0 < \alpha < 1$

For convenience let us introduce the multiscaling exponent $\tau(\rho)$ such that the partition function is

$$\langle X_a^\rho(t)\rangle \propto t^{\tau(\rho)}.$$

Results of Section 2 and the comments above immediately give that in the case of one-sided $STLF_\alpha$ flights, $0 < \alpha < 1$,

$$\log\langle |X_a(t)|^\rho\rangle \sim \tau(\rho)\log(at) + \text{const},$$

where, for $a \to 0$,

$$\tau(\rho) = \min\left(\frac{\rho}{\alpha}, 1\right),$$

and, for $a \to \infty$,

$$\tau(\rho) = \rho.$$

The simulations provided below give information about this asymptotic behavior.

In Fig. 1 we show the multiscaling exponents $\tau(\rho)$ for $STL_\alpha(a, 0, 10^{-5})$ flights, for time-scale parameter values $a = 0.0004$, and 800, and the index $\alpha = 1/2$ (top) and $\alpha = 3/4$ (bottom).

The “solid” lines represent the limiting Gaussian (with drift) case; the “dashed” lines correspond to the $\alpha$-stable processes. The “triangle-down” data points are obtained, for $a = 800$, using theoretical values of the moments. The “triangle-up” data points are obtained by simulation of $STL_\alpha(800, 0, 10^{-5})$ flights (the nontrivial simulation procedure is described in Appendix A) and estimation of $\tau(\rho)$. The two “squares” points are obtained, for $a = 0.0004$, by calculating the exact values of the moments of integer order. The “asterisk” data points are obtained by simulation of $STL_\alpha(0.0004, 0, 10^{-5})$ flights and estimation of $\tau(\rho)$.

Additionally, in the case $\alpha = 1/2$ (top), we are able to find numerically values of all fractional moments and use this information to calculate $\tau(\rho)$. The “dash-point-dash” line corresponds to $a = 800$, and “dotted” line corresponds to $a = 0.0004$.

For small time-scale parameter value $a = 0.0004$, the behavior of the multiscaling exponent $\tau(\rho)$, $\rho < \alpha$ is close to that of the $\alpha$-stable process; the smaller $\rho$ the better the approximation.

If $a = 800$, for the range of $\rho$ covered by our estimations, the situation is close to that for a Gaussian process with drift.

Note that in the case $\alpha = 1/2$ the probability density is explicitly known and we have available exact numerical evaluation of the fractional and integer order moments. So our estimations of $\tau(\rho)$ are more accurate here. We used them as a validation tool for our simulations.

### 4.2. Symmetric $STLF_\alpha$ flights, $0 < \alpha < 2$

Results of Section 3 and comments at the beginning of this section give that in the case of symmetric $STLF_\alpha$ flights, $0 < \alpha < 2$,

$$\log\langle |X_a(t)|^\rho\rangle \sim \tau(\rho)\log(at) + \text{const},$$

where, for $a \to 0$,

$$\tau(\rho) = \min\left(\frac{\rho}{\alpha}, 1\right),$$

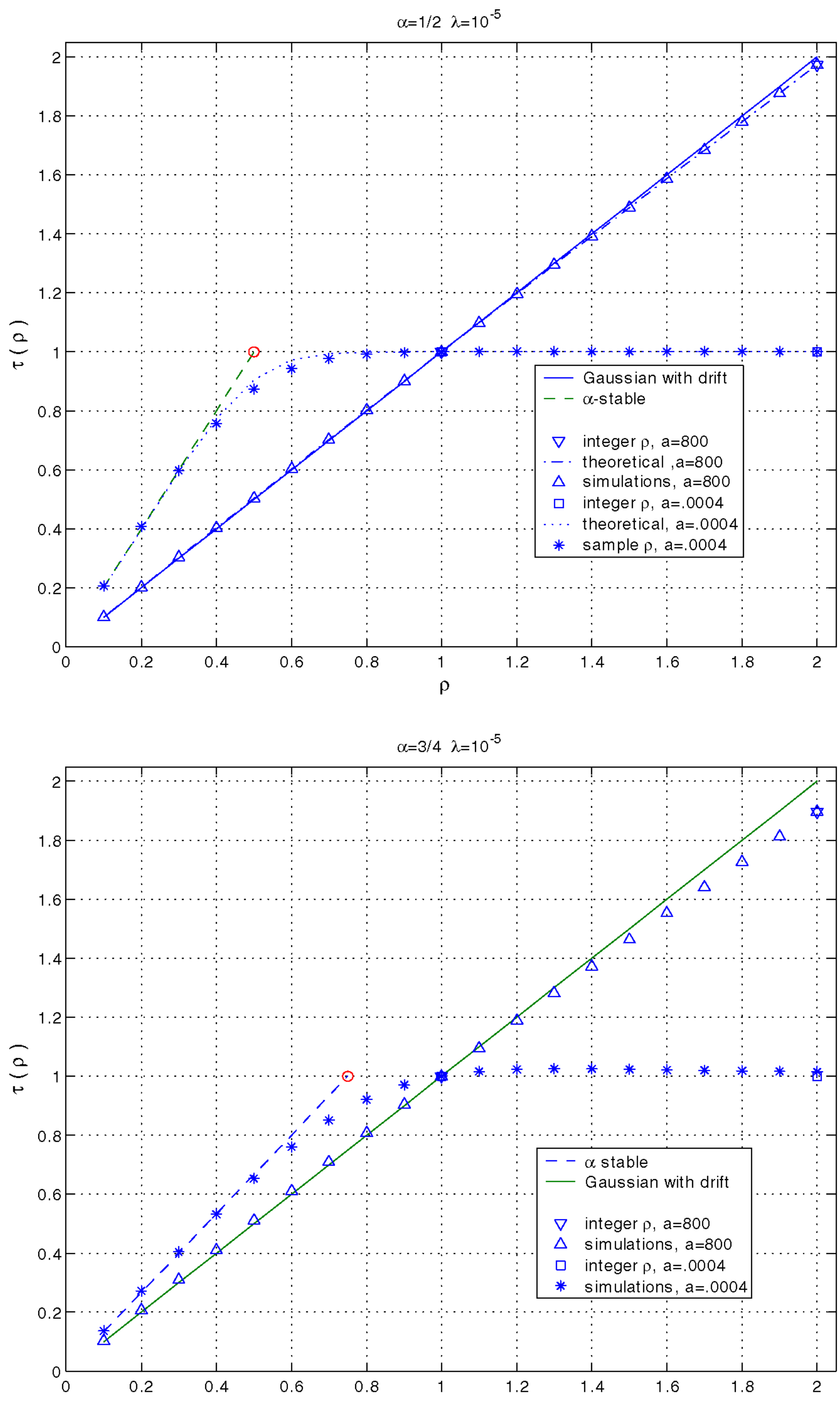


Fig. 1. The multiscaling exponents $\tau(\rho)$ for one-sided $STLF_\alpha(a, 0, 10^{-5})$, for time-scale parameter values $a = 0.0004$, and 800, and the index $\alpha = 1/2$ (top) and $\alpha = 3/4$ (bottom).

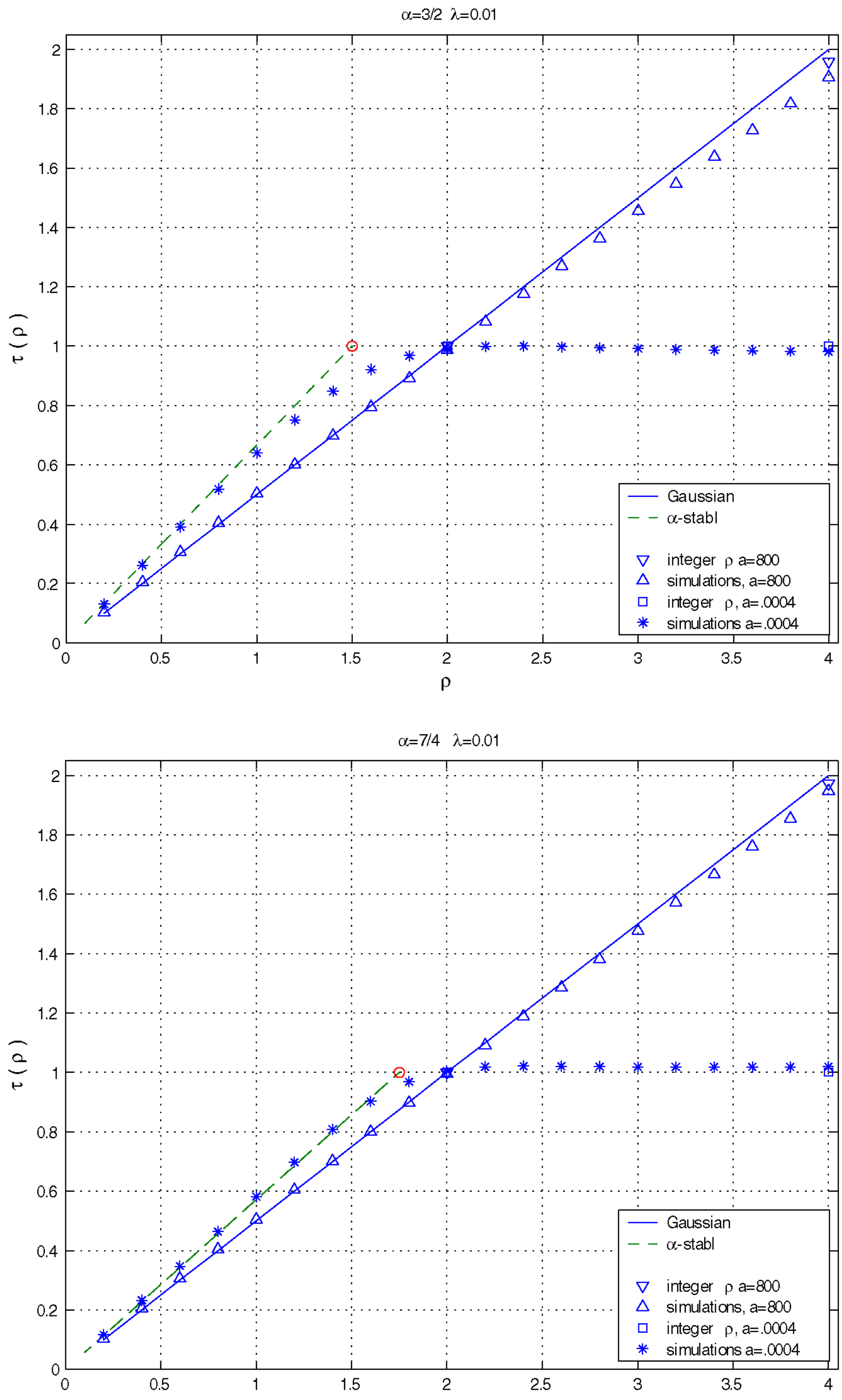


Fig. 2. The multiscaling exponents $\tau(\rho)$ for symmetric $STLF_\alpha(a, 1/2, 10^{-2})$, for time-scale parameter values $a = 0.0004$, and 800, and the index $\alpha = 3/2$ (top) and $\alpha = 7/4$ (bottom).

and, for $a \to \infty$,

$$\tau(\rho) = \frac{\rho}{2}.$$

The simulations provided below give information about this asymptotic behavior for intermediate values of $a$ and values of $\rho$ not accounted for above.

In Fig. 2 we show the multiscaling exponents $\tau(\rho)$ for symmetric $STLF_\alpha(a, 1/2, 10^{-2})$ flights, for time-scale parameter values $a = 0.0004$, and 800, and the index $\alpha = 3/2$ (top) and $\alpha = 7/4$ (bottom). The usage of special lines in this figure is identical to that of Fig. 1 which was explained in Section 4.1. For small time-scale parameter value $a = 0.0004$, the behavior of the multiscaling exponent $\tau(\rho)$, $\rho < \alpha$ is close to that of the $\alpha$-stable process; the smaller $\rho$ the better the approximation.

If $a = 800$, for the range of $\rho$ covered by our estimations, the situation is close to that for a Gaussian process with no drift.

## 5. Discussion and conclusions

The heart of the theory of $\alpha$-stable distributions is their self-similarity which permits study of their properties based on the characteristic and cumulant functions despite the fact that their densities are not known explicitly in the general case. Koponen's generalization of Mantegna and Stanley's truncation approach respects the spirit of the original theory of $\alpha$-stable distributions: the nice analytic form of the characteristic function is preserved. Further generalizations of Mantegna and Stanley's truncation scheme, such as GTLF and EDLF, take us one step further by truncating stable densities with multipliers permitting several parameters. This gives us better flexibility in fitting experimental data while taking care of the problem of infinite second moments. Of course, the price of such a flexible approach is that it takes us further away from the fundamental self-similar nature of the $\alpha$-stable distribution.

In this context, in addition to GTLF and EDLF, one can consider further possible generalization of STLFs that do retain some self-similarity characteristics. Here is one possibility: let $M(d\vartheta) = g(\vartheta)\,d\vartheta$ be a finite measure on $[0, 2\pi]$ and $\lambda(\vartheta)$ a positive function. Define the cumulant function

$$\psi_Y(u) = \int_0^{2\pi} \lambda(\vartheta)^\alpha \zeta_\alpha\left(\frac{\gamma\cos(\vartheta - \theta_1)}{\lambda(\vartheta)}u\right) g(\vartheta)\,d\vartheta, \tag{15}$$

where $\gamma > 0$ and $\theta_1 \in [0, 2\pi]$ are real numbers. It is readily seen that putting $\theta_1 = 0$, $\lambda(0) = \lambda(\pi) = \lambda$, $\gamma = 1$, and $g(0) = aq\delta_0$, $g(\pi) = ap\delta_\pi$ ($\delta_\lambda$ is the Dirac measure), we obtain the cumulant function

$$\psi_{X(1)}(u) = a\lambda^\alpha\big[p\zeta_\alpha(-u/\lambda) + q\zeta_\alpha(u/\lambda)\big],$$

of $STLF_\alpha(at, p, \lambda)$ as a particular case. Let us call the model based on formula (15), generalized smoothly truncated Lévy flights (GSTLF). The model is very flexible as it admits not only numerical parameters $\alpha, \theta_1$ and $\gamma$, but also functional parameters $\lambda(\vartheta)$ and $g(\vartheta)$. For practical purposes these functions can be chosen to depend on finitely many parameters. Note that this method can be also robustly applied to GSTLF. For an even larger family of generalizations, see [12].

In this Letter we have shown that, for a one-sided $STLF_\alpha(a, 0, \lambda)$ distribution, the moments of fractional positive order $\rho$ has the asymptotics

$$\log\langle|X|^\rho\rangle \sim \begin{cases} \min(\rho/\alpha, 1)\log a + c_1, & \text{as } a \to 0, \\ \rho\log a + c_2, & \text{as } a \to \infty, \end{cases}$$

and for the symmetric distribution $STLF_\alpha(a, 1/2, \lambda)$

$$\log\langle |X|^\rho\rangle \sim \begin{cases} (\rho/\alpha)\log a + C_1, & \text{for } a \to 0, \\ (\rho/2)\log a + C_2, & \text{for } a \to \infty. \end{cases}$$

The consequence of these result is the multiscaling properties of the STFLs.

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## Appendix A

### A.1. *Derivation of the scaling exponents for symmetric $STLF_\alpha$, $0 < \alpha < 2$*

Let us begin by recalling that for any random quantity $Z$, with characteristic function $\varphi_Z(u)$, fractional moments are given by the formula

$$\langle |Z|^\rho\rangle = c(\rho)\int_{-\infty}^{\infty}\bigl[1 - \operatorname{Re}\varphi_Z(u)\bigr]\frac{du}{|u|^{1+\rho}}, \tag{A.1}$$

where $0 < \rho < 2$, and

$$c(\rho) = \frac{\Gamma(1+\rho)\sin(\rho\pi/2)}{\pi},$$

see [16].

#### A.1.1. *The case $0 < \alpha < 2$, $\alpha \neq 1$*

Denote $\theta_y = \arctan y$ and substitute the characteristic function $\varphi(u) = \exp(\psi(u))$, with $\psi(u)$ given by the first formula in (12), into formula (A.1). In view of the symmetry of the distribution, for $\rho < \alpha$,

$$\langle |X|^\rho\rangle = 2c(\rho)\int_0^\infty 1 - \exp\bigl(-a\lambda^\alpha\Gamma(-\alpha)\bigl[1 - (1 + u/\lambda)^{\alpha/2}\cos(\alpha\theta_{u/\alpha})\bigr]\bigr)\frac{du}{u^{1+\rho}}. \tag{A.2}$$

Integration by parts, and change of variable $y = u/\lambda$, yields the following expression:

$$\begin{aligned}\langle |X|^\rho\rangle = {}& \frac{2c(\rho)}{\rho}a\lambda^{\alpha-\rho}\int_0^\infty \exp\bigl[-a\lambda^\alpha\Gamma(-\alpha)\bigl(1 - \bigl(1+y^2\bigr)\cos\alpha\theta_y\bigr)\bigr] \\ & \times \bigl(1+y^2\bigr)^{\alpha/2-1}\Gamma(-\alpha)\biggl(\frac{1}{y}\sin(\alpha\theta_y) - \cos(\alpha\theta_y)\biggr)\frac{dy}{y^{\rho-1}}.\end{aligned} \tag{A.3}$$

Since the asymptotics of this integral, both, for small and large time-scale $a$, is difficult to evaluate, we begin by finding first upper and lower bounds for the integrands by using obvious inequalities

$$\Gamma(-\alpha)(\alpha-1) \leqslant \frac{\Gamma(-\alpha)(1 - (1+y^2)^{\alpha/2}\cos(\alpha\theta_y))}{(1+y^2)^{\alpha/2} - 1} \leqslant -\Gamma(-\alpha)\cos\alpha\pi/2, \tag{A.4}$$

$$\Gamma(-\alpha)(\alpha-1)\leqslant\Gamma(-\alpha)\left(\frac{1}{y}\sin(\alpha\theta_y)-\cos(\alpha\theta_y)\right)\leqslant-\Gamma(-\alpha)\cos(\alpha\pi/2). \tag{A.5}$$

*The upper bound.* Applying right-hand sides of inequalities (A.4) and (A.5) to formula (A.3) we have

$$\begin{aligned}\langle|X|^\rho\rangle\leqslant{}&\frac{2c(\rho)}{\rho}a\lambda^{\alpha-\rho}\\&\times\int_0^\infty\exp\bigl[-a\lambda^\alpha\Gamma(-\alpha)(\alpha-1)\bigl(\bigl(1+y^2\bigr)-1\bigr)\bigr]\bigl(1+y^2\bigr)^{\alpha/2-1}\bigl(-\Gamma(-\alpha)\cos(\alpha\pi/2)\bigr)\frac{dy}{y^{\rho-1}}.\end{aligned}$$

Denote

$$\begin{aligned}&A=-2c(\rho)a\lambda^{\alpha-\rho}\Gamma(-\alpha)\cos(\alpha\pi/2)/\rho,\\&b=a\lambda^\alpha\Gamma(-\alpha)(\alpha-1)\end{aligned}$$

and observe that, for $0<\alpha<1$, and, for $1<\alpha<2$, constant $b$ is positive. Thus we get that

$$\begin{aligned}\langle|X|^\rho\rangle&\leqslant A\int_0^\infty\exp\bigl[-b\bigl(\bigl(1+y^2\bigr)^{\alpha/2}-1\bigr)\bigr]\bigl(1+y^2\bigr)^{\alpha/2-1}\frac{dy}{y^{\rho-1}}=\frac{A}{\alpha}\int_1^\infty\exp\bigl(-b(y-1)\bigr)\bigl(y^{2/\alpha}-1\bigr)^{-\rho/2}dy\\&=\frac{A}{\alpha}\int_1^\infty\frac{\exp(-b(y-1))}{(y-1)^{\rho/2}}\left(\frac{y-1}{y^{2/\alpha}-1}\right)^{\rho/2}dy.\end{aligned}$$

Now, function $f(y):=(y-1)/(y^{2/\alpha}-1)$ can be expanded in power series around $y=1$, and $y=\infty$, in a procedure similar to the one used in Section 2, with $1/\alpha$ replaced by $2/\alpha$.

If $a\to0$, or, equivalently, $b\to0$, then

$$\begin{aligned}\langle|X|^\rho\rangle&\leqslant\frac{A}{\alpha}\int_1^\infty\frac{\exp(-b(y-1))}{(y-1)^{\rho/2}}\left(\frac{y-1}{y^{2/\alpha-1}}\right)^{\rho/2}=\frac{A}{b\alpha}\int_0^\infty\frac{\exp(-u)}{(u/b)^{\rho/\alpha}}\left(\frac{u/b}{(u/b+1)^{2/\alpha}-1}\right)^{\rho/2}du\\&=\frac{A}{b\alpha}\int_0^\infty\frac{\exp(-u)}{(u/b)^{\rho/\alpha}}\left(\frac{b^{2/\alpha-1}}{u^{2/\alpha-1}}+o\left(\frac{1}{(u/b)^{2/\alpha}}\right)\right)du\sim\frac{A}{b\alpha}b^{\rho/\alpha}\Gamma(1-\rho/\alpha)\\&=a^{\rho/\alpha}\frac{2\sin(\rho\pi/2)\cos(\alpha\pi/2)}{\pi\alpha(1-\alpha)}\bigl(\Gamma(-\alpha)(\alpha-1)\bigr)^{\rho/\alpha}\Gamma(\rho)\Gamma(1-\rho/\alpha).\end{aligned}$$

Therefore, there exists a positive constant $c_1$ such that, for sufficiently small $a$,

$$\log\langle|X|^\rho\rangle\leqslant\frac{\rho}{\alpha}\log a+c_1.$$

If $a\to\infty$, or equivalently, $b\to\infty$, and $u/b\to0$, then

$$\begin{aligned}\langle|X|^\rho\rangle&\leqslant\frac{A}{b\alpha}\int_0^\infty\frac{\exp(-u)}{(u/b)^{\rho/\alpha}}\left(\frac{u/b}{(u/b+1)^{2/\alpha}-1}\right)^{\rho/2}du\sim\frac{A}{b\alpha}\int_0^\infty\frac{\exp(-u)}{(u/b)^{\rho/\alpha}}\left(\frac{\alpha}{2}\right)^{\rho/2}du\\&=\frac{A}{b\alpha}\left(\frac{\alpha b}{2}\right)^{\rho/2}\Gamma(1-\rho/2)=a^{\rho/2}\frac{2\sin(\pi/2)\cos(\alpha\pi/2)}{\pi\alpha(1-\alpha)}\left(\frac{\lambda^{\alpha-2}\Gamma(2-\alpha)}{2}\right)^{\rho/2}\Gamma(\rho)\Gamma(1-\rho/2).\end{aligned}$$

Therefore, there exists a positive constant $c_2$ such that, for sufficiently large $a$,

$$\log\langle |X|^\rho\rangle \leqslant \frac{\rho}{2}\log a + c_2.$$

*The lower bounds.* Here we proceed in a manner analogous to the estimates used to obtain the upper bounds but employing the complementary inequalities in (A.4) and (A.5). The details are omitted. The final result is that there exist constants $c_3$ and $c_4$, independent of $a$, such that, for small enough $a$,

$$\log\langle |X|^\rho\rangle \geqslant \frac{\rho}{\alpha}\log a + c_3$$

and, for sufficiently large time-scale parameter $a$,

$$\log\langle |X|^\rho\rangle \geqslant \frac{\rho}{2}\log a + c_4.$$

*A.1.2. The case $\alpha = 1$*

Note that the cumulant function $\psi_{X_a}(u)$ in (12), for $\alpha = 1$, is the limit of the cumulant functions $\psi_{X_a}(u)$ as $\alpha \searrow 1, \alpha \neq 1$, and the same is true for the corresponding characteristic functions. Therefore, for any $\rho < 1$, the integrand in (A.2) for $\alpha = 1$ is the bounded limit of the integrands in (A.2) for $\alpha \searrow 1, \alpha \neq 1$. As a result for $\alpha = 1$ we get the asymptotics

$$\log\langle |X|^\rho\rangle \sim \begin{cases} \rho\log a + C_3, & \text{for } a \to 0, \\ (\rho/2)\log a + C_4, & \text{for } a \to \infty, \end{cases}$$

for some, independent of $a$, constants $C_3$, $C_4$.

*A.2. Monte Carlo method for simulation of STLFs*

For our simulations of STLF random variables we have used the series representations obtained in Rosiński [12]. In the one-sided $STLF_\alpha(a, 0, \lambda)$ case, with $0 < \alpha < 1$, the expansion is as follows:

$$S_0 = \sum_{j=1}^{\infty}\left(\left(\frac{\alpha\gamma_j}{a}\right)^{-1/\alpha} \wedge \frac{e_j u_j^{1/\alpha}}{\lambda}\right),$$

where $\{u_j\}$ is a sequence of independent and identically distributed (iid) uniform random variables on the interval $[0, 1]$, $\{e_j\}$ and $\{e'_j\}$ sequences of iid exponential random variables with parameter 1, and $\gamma_j = e'_1 + \ldots + e'_j$. Additionally, the sequences $\{u_j\}$, $\{e_j\}$ and $\{e'_j\}$ must be independent of each other. The wedge $\wedge$ stands for the operation of taking the minimum of two real numbers.

In the symmetric $STLF_\alpha(a, 1/2, \lambda)$ case the expansion is

$$S_1 = \sum_{j=1}^{\infty}\left(v_j\left(\frac{\alpha\gamma_j}{a}\right)^{-1/\alpha} \wedge \frac{e_j u_j^{1/\alpha}}{\lambda}\right),$$

where the additional sequence $\{v_j\}$ sequence appearing here consists of independent random variables taking values $\pm 1$ with equal probabilities $1/2$.

The $STLF_\alpha(a, p, \lambda)$ flight at an integer time $t = 2, 3, \ldots,$ is simulated as a sum of iid copies of the same flight at time 1.

In the one sided-case $p = 0$, we choose $\lambda = 10^{-5}$, and $a = 0.0004$ or $= 800$. In the symmetric case, we choose $\lambda = 10^{-2}$, and $a = 0.0004$ or $= 800$.

To obtain the multiscaling exponent $\tau(\rho)$, we take into account the fact that, for small and large $a$ the relationship between $\log\langle |X_a(t)|^\rho\rangle$ and $\log a$, is nearly linear, choose times exponentially $t = 1, 2, 2^2, \ldots, 2^5$, and estimate the slope via linear regression.

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